Chapter 5

BONDS

I) Introduction

A bond is a part of the debt of an institution. It has:

- A face value or nominal value N
- An annual nominal interest rate z
- Each year, the interest paid is the coupon $C = z * N$
- An emission price E:
    - If $E = N$, the bond is emitted "at par"
    - If $E < N$, the bond is emitted "at a discount"
    - If $E > N$, the bond is emitted "at a premium"
- A refund (or repay) price R:
    - If $R = N$, the refund is "at par"
    - If $R < N$, the refund is "at a discount"
    - If $R > N$, the refund is "at a premium"

A bond can be exchanged many times on the secondary market between its date of issue and its date of maturity. Its price varies in an unpredictable way along this period.

Question: How can one choose between different bonds?

Example: Take two bonds A end B each of maturity 2 years, same nominal $N_A = N_B = 1000$. Assume that A delivers two coupons at the end of years 1, 2, each of value $C_A = 100$.
Assume that B delivers only 1 coupon of $C_B = 1000$ at the end of year 1.
Assume that both A, B are refunded at par: $R_A = R_B = N = 1000$.

Time	1	2
A	100	1100
B	1000	1000

Assume that A is emitted at $E_A = 1000$, and B at $E_B = 1735$.

Comparison of A and B

1st method: choose an interest rate r, considered as reference (for instante, the interest rate of American Treasury Bonds)

Then $NPV(A) = -1000 + \frac{100}{1+r} + \frac{1100}{(1+r)^2} = -E_A + \frac{C_A}{1+r} + \frac{C_A + R_A}{(1+r)^2}$

$$NPV(B) = -1735 + \frac{1000}{1+r} + \frac{1000}{(1+r)^2}$$

If $NPV(A) > NPV(B)$, you should prefer A to B
If $NPV(A) < NPV(B)$, you should prefer B to A
If $NPV(A) = NPV(B)$, A and B are indifferent

II) The arbitrage price of a bond

(we can also say the price of no arbitrage of a bond)

Definition: an arbitrage at time t is a set of financial operations, corresponding to cash flows that add to zero at time t, and which will necessarily generate non-negative cash flows in the future, with a positive probability that at least one of the future cash flows will be positive.

In an ideal market, arbitrage is impossible. This means that the market fixes the prices of assets in such a way that no arbitrage can occur.
Our goal is to compute the value of bond under the no arbitrage hypothesis.

A) Simplified (and unrealistic) case

Take a market with only two kinds of securities:
- A riskless bond A (i.e. American Treasury Bonds) at a fixed interest rate r
- A (possibly risky) bond B that generates future cash flows at time $t+1, \dots, t+m$ denoted F_{t+i}

Claim: the arbitrage price of bond B at time t is

$$P_{arbitrage}(B) = \sum_{i=1}^{n} \frac{F_{t+i}}{(1+r)^i}$$

In other words, $P_{arbitrage}(B)$ is such that $NPV(B,t) = 0$ where $NPV(B,t) = -P_{arbitrage}(B(t)) + \sum_{i=1}^{n} \frac{F_{t+i}}{(1+r)^i}$

Proof of the claim: we argue by contradiction. Let $P_* = \sum_{i=1}^{n} \frac{F_{t+i}}{(1+r)^i}$.
Assume that $P_t < P_*$. Then we can do as following:
- At time t, you buy one bond B at price P_t, and you sell a fraction f of Treasury Bond A, such that $Unitprice(A) * f = P_t$. (This is equivalent to borrowing the amount P_t at interest rate r)
- At time 0, your net cash flow is $-P_t + P_t = 0$.
- At time $t+i$, you plan to do the following: you receive the amount F_i and you reinvest it in Treasury bonds, for $i = t+1, \dots, t, +n-1$. → net cash flows = 0 at times $t+1, \dots, t+n-1$.
- At time $t+n$, you receive F_{t+n} and you close your account with the treasury bonds A. The amount you receive from treasury bonds at time $t+n$ is:
$$-P_t(1+r)^n + F_{t+1}(1+r)^{n-1} + \dots + F_{t+i}(1+r)^{n-i} + \dots + F_{t+n-1}(1+r)$$
So your net cash flow at time $t+n$ is:
$$-P_t(1+r)^n + \sum_{i=1}^{n-1} F_i(1+r)^{n-i} + F_{t+n} = (1+r)^n(-P_t + P_*) > 0$$
Arbitrage is possible, so $P_t \geq P_*$.
Similarly, if $P_t > P_*$, do exactly the opposite operations; you get an arbitrage, which is forbidden.

So $P_t = P_*$ in absence of arbitrage.

III) Pricing of a bond using zero-coupon bonds

Definition: a zero-coupon bond is a bond that delivers no coupon. At maturity, the owner of the zero-coupon bond receives an amount R. As a result, a zero coupon bond is characterized at time t by 3 quantities:
- Its price P_t at time t
- Its date of maturity $T > t$
- Its refund price R (payed at time T)

Example: Suppose that on the markets, there are four types of bonds A, B, C, D available at time $t = 0$

Bond	$t=0$	$t=1$	$t=2$
A	-1000	100	1100
B	-1735	1000	1000

C	-95	100	0
D	-80	0	100

C, D are zero-coupon bonds
C has price 95 at time 0, its maturity date is 1 and its referred value is 100.
D has price 80 at time 0, its maturity date is 2 and its referred value is 100.
Using the zero-coupon bonds C and D, we can "replicate" bond A.
Buy 1 unit of bond C + 11 units of bond D, at time 0. You pay $95 + 11 * 80 = 95 + 880 = 975$
At time 1, you receive: $\left(1 * \text{bond}(A) + 11 * \text{bond}(B)\right) 100 + 11 * 0 = 100$
At time 2, you receive: $1 * 0 + 11 * 100 = 1100$

Using this "replication", I can construct an arbitrage ("free lunch")

$t = 0$		$t = 1$	$t = 2$
sell 1 unit of A	+1000	-100	-1100
Buy 1 unit of C	-95	$+100$	0
Buy 11 units of D	-880	0	1100
Invest 25€ at the "caisse d'épargne"	-25		$25 * (1 + r)^{@}$
0		0	$25 * (1 + r)^{@} > 0$

Conclusion: considering C and D, the price of A at time 0 is above its no-arbitrage-price

$$P_{\mathrm{jk1mnkjop}}(A) = 975$$

Similarly, we can replicate bond B as follows:
At time 0, buy $10 * C + 10 * D$: you pay $950 + 800 = 1750$.
This is higher than the price of bond B. So we make an arbitrage:

$t = 0$	$t = 1$	$t = 2$
Buy one unit of bond B : -1735	1000	1000
Sell 10 units of C : 950	-1000	0
Sell 10 units of D : 800	0	-1000
+15	0	0

General case: on a financial market, at time t, we will admit that there are zero-coupon bonds of all maturities $T \geq t$. We will normalize them so that for each unit of such bonds, the value at maturity is $R = 100$. (Note that for a zero-coupon bond, one considers that $N = R$).
The price of the zero-coupon bond of maturity date T is $P(t, T)$ at time $t \leq T$.
One defines an interest rate per year $z(t, T)$ of this zero-coupon bond by the formula

$$R = N = \left(1 + z(t, T)\right)^{(\mathrm{r} \backslash \mathrm{H})} P_{\mathrm{H}}$$

In fact, $z(t, T)$ is the IRR of the zero-coupon bond of maturity date T.
Indeed, if you buy this bond at time t, the PV at time t of your cash flows with actualization rate $z(t, T)$ is:

$$PV_{\mathrm{t}(\mathrm{H},\mathrm{r})}(\text{zero} - \text{coupon bond}) = -P_{\mathrm{H}} + \left(1 + z(t, T)\right)^{\mathrm{H} \backslash \mathrm{r}} * N = 0$$

Then, one draws the so-called zero-rate curve at time t. Usually, the function $T \rightarrow z(t, T)$ is increasing
The curve can be flat or decreasing, but this is exceptional: it means that the market anticipates a lowering of interest rates.
The formula for $z(t, T)$ knowing $P_{\mathrm{H},\mathrm{r}}$ (price at time t of zero-coupon bond of maturity date T) T and

$N = 100$ is :

$$z(t, T) = \left(\frac{1}{3_{\mathbb{Q},\sim}}\right)^{\frac{?}{\sim q\mathbb{Q}}} - 1$$

(obtained by solving $N = \left(1 + z(t, T)^{(r \setminus \mathbb{H})} P_{\mathbb{H},r}\right)$)

Now take a bond B delivering future cash flows $F_{\mathrm{H}_?}, F_{\mathrm{H}_<}, \ldots, F_{\mathrm{H}_{\ddot{A}}}$ at future times $t_8 < t_@ < \cdots < t_\mathrm{N} = T$

It's no arbitrage price at time $t < t_8$ is $P_{\mathrm{J;K}}(B) = \sum_{\mathrm{IO8}}^{\mathrm{N}} \frac{\mathrm{P}_{\mathrm{Q}_\mathrm{S}}}{(8 : \mathrm{t}(\mathrm{H}, \mathrm{H}_\mathrm{S}))^{\mathrm{Q}_\mathrm{S}\mathrm{qQ}}}$