Chapter 3

Loans

A loan takes place between two agents.

$$A = borrower$$
$$B = lender$$

I) Notations

$$C = capital: \text{the amount borrowed by } A \text{ at } t = 0$$
$$(\text{sometimes } C \text{ will be denoted } C_0 \text{ or } V_0 \text{ for value at time } 0)$$

$$n = \text{maturity of the loan (usually in years)}:$$
$$\text{time delay between time } 0 \text{ (beginning of the loan) and the last payment made by } A$$

$$r = \text{nominal annual rate of the loan}$$

$$I_k = \text{amount of interests paid by } A \text{ during year } k$$
$$(\text{usually } I_k \text{ is paid at the end of year } k)$$

$$A_k = \text{part of capital refunded during year } k \text{ (amortization)}$$

$$E_k = I_k + A_k : \text{annuity paid by } A$$

$$C_k \text{ or } V_k = \text{capital due at the end of year } k$$
$$(V_k = V_{k-1} - A_k)$$

At the end of year n, we must have $V_n = 0$, but

$$V_n = V_{n-1} - A_n = V_{n-2} - A_{n-1} - A_n = \cdots = C - A_1 - \cdots - A_n$$

So $C = A_1 + A_2 + \cdots + A_n$

If we assume that the period of compounding is 1 year then r is also the effective annual rate. Then

$$I_k = rV_{k-1} \qquad (\text{with } C = V_0)$$

The total cost of the loan is $cost = \sum_{k=1}^{n} I_k$

II) Repayment methods

1) Repayment of capital "in fine"

We assume that payments of annuities occur at the end of each year: times $1, 2, \ldots, n$

This means that $\begin{cases} A_k = 0 \text{ for } k = 1,2,\ldots,n-1 \\ A_n = C = V_0 \end{cases}$

Then $V_k = V_0 = C$ for $k = 0,1,\ldots,n-1$ and $V_n = 0$ (as always)

$$I_1 = I_2 = \cdots = I_n = rC = rV_0$$

We can summarize this in an amortization table:

k	V_k	I_k	A_k	E_k
0	V_0	0	0	0
1	V_0	rV_0	0	rV_0

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$V_{:}$	$rV_{:}$	0	$rV_{:}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$V_{:}$	$rV_{:}$	0	$rV_{:}$
n	0	$rV_{:}$	$V_{:}$	$1+r\ V_{:}$
$TOTAL$		$cost = nrV_{:}$	$V_{:}$	$1+nr\ V_{:}$

This means that $A_0 = 0$, $A_1 = A_2 = \cdots = A_n$

So $V_0 = nA$ $\quad\quad$ $A = \frac{V_0}{n}$

So $V_k = V_{k-1} - A$ $\quad\quad$ V_k is an arithmetic sequence

So $V_k = V_0 - k * A = \left(1 - \frac{k}{n}\right) V_0$

So $I_k = rV_{k-1} = r(V_0 - kA) = r\left(1 - \frac{k-1}{n} V_0\right)$

$$E_k = A + I_k = \left(\frac{1}{n} + \frac{r(n-k+1)}{n}\right) V_0 = \frac{(1 + (n-k+1)r)}{n} V_0$$

k	V_k	I_k	A_k	E_k
0	V_0	0	0	0
1	$\left(1 - \frac{1}{n}\right) V_0$	rV_0	$\frac{V_0}{n}$	$\frac{1 + nr}{n} V_0$
⋮	⋮	⋮	⋮	⋮
k	$\left(1 - \frac{k}{n}\right) V_0$	$r\left(1 - \frac{k-1}{n}\right) V_0$	$\frac{V_0}{n}$	$\frac{1 + (n-k+1) r}{n} V_0$
⋮	⋮	⋮	⋮	⋮
$n - 1$	$\frac{V_0}{n}$	$\frac{2r}{n} V_0$	$\frac{V_0}{n}$	$\frac{1 + 2r}{n} V_0$
n	0	$\frac{r}{n} V_0$	$\frac{V_0}{n}$	$\frac{(1 + r)}{n} V_0$
$TOTAL$		$cost = \frac{n+1}{2} rV_0$	V_0	$V_0 + cost$

At time 0, an initial capital is borrowed, its amount is denoted K or V_0.

Period	Interest	Amortization	Annuity	Outstanding loan capital
1	I_1	A_1	a_1	V_1
⋮	⋮	⋮	⋮	⋮
k	I_k	A_k	a_k	V_k
⋮	⋮	⋮	⋮	⋮
n	I_n	A_n	a_n	0

2) Method of constant annuity

We borrow an amount of capital $K = V_0$ at time 0, at an annual rate $r > 0$ (one can even allow $r > -1$). Repayment will take place at the end of each year, and annuity will be constant : $a_1 = a_2 = \cdots = a_n = a$.

Question: find the value of a, using the method of present values.
Suppose we borrow $V_0 = K$ and that annuities are $a_1, \ldots, a_n$.
At the end of period n, $I_n = r * V_{n-1}$ $\quad\quad$ $a_n = I_n + A_n$,
$A_n = V_{n-1}$ $\quad$ so $\quad$ $a_n = r * V_{n-1} + V_{n-1} = (1 + r)V_{n-1}$
So

$\underbrace{V_{n-1}}$ $\quad\quad$ $\underbrace{(1 + r)^{-1}}$

Similarly, at the end of period $k+1$, $(1 \le k \le n-1)$

$$I_{k+1} = rV_k$$

$$a_{k+1} = rV_k + V_k - V_{k+1} = (1+r)V_k - V_{k+1}$$

$$V_k = (1+r)^{-1}\left(a_{k+1} + V_{k+1}\right)$$

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$$= (1+r)^{FG} a_{@mG} + (1-r)^{FG} V_{@mG}$$

$$= (1+r)^{FG} a_{@mG} + (1+r)^{FG}\left((1+r)^{FG} a_{@mJ} + (1+r)^{FG} V_{@mJ}\right)$$

$$= (1+r)^{FG} a_{@mG} + (1+r)^{FJ} a_{@mJ} + (1+r)^{FJ} V_{@mJ}$$

$$= (1+r)^{FG} a_{@mG} + (1+r)^{FJ} a_{@mJ} + \cdots + (1+r)^{FIm@mG} a_{IFG} + (1+r)^{IF@mG} V_{IFG}$$

$$\underbrace{V_@}_{Z[\backslash].\ \_`a.\ `\wedge\wedge fg\backslash\ @} = (1+r)^{FG} \underbrace{a_{@mG}}_{db\wedge.\ h`]e\ dacj\ `\wedge\wedge fg\backslash\ @mG} + (1+r)^{FJ} \underbrace{a_{@mJ}}_{db\wedge.\ h.d\ `\wedge\ @mJ} + \cdots + (1+r)^{F(IF@FG)} \underbrace{a_{IFG}}_{db\wedge.\ h.d\ `\wedge\ IFG} + (1+r)^{F(IF@} a_I$$

In particular:

$$K = V_: = (1+r)^{FG} a_G + (1+r)^{FJ} a_J + \cdots + (1+r)^{FI} a_I = \sum_{fLG}^{I} (1+r)^{Ff} a_f$$

<u>Application to constant annuity:</u>

$$a_G = a_J = \cdots = a_I = a$$

So

$$K = \left(\sum_{fLG}^{I} (1+r)^{Ff}\right) a = \frac{1}{1+r} \sum_{oL:}^{IFG} \left(\frac{1}{1+r}\right)^{o} a = \frac{1}{1+r} * \frac{1-\left(\frac{1}{1+r}\right)^{I}}{1 - \frac{1}{1+r}} a$$

$$= \frac{1-(1+r)^{FI}}{r} a$$

$$\Rightarrow a = \frac{rK}{1-(1+r)^{FI}}$$

Now, for $k \in [\![1;n]\!]$, $I_@ = rV_{@F1}$, $a = I_@ + A_@$

So $A_@ = a - rV_{@FG}$

$$V_@ = \sum_{fLG}^{IF@} (1+r)^{Ff} a_{@mf} = \left(\sum_{fLG}^{IF@} (1+r)^{Ff}\right) a = \frac{1-(1+r)^{F(IF@}}{r} a$$

Application to constant annuity:

$$V_@ = \frac{1-(1+r)^{F(\quad)}}{1-(1+r)} K$$

$$A_@ = V_{@FG} - V_@ = \frac{(1+r)^{F\,IF@}(-1+(1+r)^{F})}{(\ )^{IF@mG}} K$$

$$= \frac{(1+r)^{F(\quad)}((1+r)-1)}{1-(1+r)} K = (1+r)^{@FG} \frac{rk}{(1+r)^{I} -}$$

So

$$A_@ = (1+r)^{@FG} A_G$$

with $A_G = \frac{[s}{(Gm[)}$.

We have proved that $(A_@)$ is a geometric sequence of multiplier $(1+r) > 1$, if $r > 0$.

What is the total cost of the loan?

$$Cost = \sum_{@LG}^{I} I_@ = \sum_{@LG}^{I} (a - A_@) = na - \sum_{@LG}^{I} A_@ = na - K$$

$$\Rightarrow cost = \frac{nrK}{1-(1+r)} - K = \left(\frac{nr}{1-(1+r)} - 1\right) K$$

Comment: if you have a loan with monthly payment with annual nominal rate r, this means that compounding is applied, with compounding period of a month or a twelfth a year.

The monthly interest rate is $r_{g} = \frac{[}{G}$ (since r is nominal)

Now if the capital borrowed is K and the monthly annuities are $a_{G}, \ldots, a_{u}$ (months $1, 2, \ldots, N$) $N = 36$ for 3 years.

We must have $K = \sum_{fLG}^{u} (1 + r_{g})^{Ff} a_{f}$